Gravitation as a Supersymmetric Gauge Theory

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Abstract

We propose a gauge theory of gravitation. The gauge potential is a connection of the Super SL(2,C) group. A MacDowell-Mansouri type of action is proposed where the action is quadratic in the Super SL(2,C) curvature and depends purely on gauge connection. By breaking the symmetry of the Super SL(2,C) topological gauge theory to SL(2,C), a spinor metric is naturally defined. With an auxiliary anti-commuting spinor field, the theory is reduced to general relativity. The Hamiltonian variables are related to the ones given by Ashtekar. The auxiliary spinor field plays the role of Witten spinor in the positive energy proof for gravitation.

It was Einstein's great insight that gravity is a manifestation of space-time curvature. In Riemannian geometry, a space-time with curvature is described by the metric and Einstein therefore used it to describe gravity.

The prime principle of theoretical physics is, however, to begin with the action. It gives consistent interacting field equations with the desired conserved quantities. Moreover all other basic interactions have a variational foundation. An obvious candidate for the gravitational variables for the action is the metric (or the tetrad). However, nature may prefer other candidates. In view of the success of gauge theory, the question of whether gravity can be written in terms of a pure connection theory has been addressed [1].

Important progress was made with the Ashtekar's New Variables [2], in which a self-dual connection is the basic Hamiltonian variable. The momentum conjugate to the self-dual connection is the "triad" made from a self-dual 2-form. Cappovilla, Dell and Jacobson then proposed a purely connection action for the New Variables [3].

Recently, a quadratic spinor Lagrangian for general relativity was proposed, where a Dirac spinor 1-form field [4] (or two 2-component spinor 1-form fields [5]) was considered as the variational variable. (Another 2-form version of the Lagrangian was also discovered by Robinson [6]). The space-time metric is represented by a quadratic combination of the spinor 1-form field. This provides a succinct representation for general relativity, and provides a way to represent the gravitational energy-momentum [4]. With a reality condition satisfied, the quadratic spinor Lagrangian is equivalent to the teleparallel Lagrangian for general relativity [7]. Robinson discussed the theory using a Lie algebra [8].

As the variable has half helicity, one may ask if it can be an anti-commuting field. Jacobson suggested using a single anti-commuting spinor 1-form [9]; Robinson provided a formulation with two anti-commuting spinor 1-forms [6]. Recently we learned that another similar anti-commuting approach was used long ago in the context of supergravity [10].

The purpose of this letter is to propose a gauge theory along these lines. We start from a gauge theory of the Super SL(2,C) group, giving a topological gauge theory of the MacDowell-Mansouri type. By breaking this supersymmetry into the SL(2,C) symmetry, we get two equivalent chiral actions. We show that, up to a nilpotent metric, the theory is equivalent to general relativity.

Let us start with a Super SL(2,C) algebra (with three SL(2,C) generators

$M_{00}, M_{01} = M_{10}, M_{11}$ and two supersymmetric generators Q_0, Q_1):

$$
\begin{aligned}
[M_{AB}, M_{CD}] &= \epsilon_{C(A} M_{B)D} + \epsilon_{D(A} M_{B)C}, && (1)\\
[M_{AB}, Q_C] &= \epsilon_{C(A} Q_{B)}, && (2)\\
\{Q_A, Q_B\} &= 2M_{AB}, && (3)
\end{aligned}
$$

where $\epsilon_{C(A} M_{B)D} = \frac{1}{2}(\epsilon_{CA} M_{BD} + \epsilon_{CB} M_{AD})$ and $\epsilon_{C(A} Q_{B)} = \frac{1}{2}(\epsilon_{CA} Q_B + \epsilon_{CB} Q_A)$.

The upper-case Latin letters $A, B, ... = 0, 1$ denote two component spinor indices, which are raised and lowered with the constant symplectic spinors $\epsilon_{AB} = -\epsilon_{BA}$ together with its inverse and their conjugates according to the conventions $\epsilon_{01} = \epsilon^{01} = +1$, $\lambda^A := \epsilon^{AB} \lambda_B$, $\mu_B := \mu^A \epsilon_{AB}$ [11]. Lowercase Latin letters $p, q, ...$ denote the Super SL(2,C) group indices, $a, b, c, ... = 0, 1, 2, 3$ denote the SO(3,1) indices .

The Super SL(2,C) group is isomorphic to the complex extension of OSp(1,2). It is a *simple* super Lie group and has a nondegenerate Killing form [12]. The Cartan-Killing metric $\eta_{pq} = \mathrm{diag}(\eta_{(AB)(MN)}, \eta_{AB})$ is given by

$$
\begin{aligned}
\eta_{(AB)(MN)} &= \frac{1}{2}(\epsilon_{AM} \epsilon_{BN} + \epsilon_{AN} \epsilon_{BM}), && (4)\\
\eta_{AB} &= -2\epsilon_{AB}. && (5)
\end{aligned}
$$

To *gauge* this Super SL(2,C) group, we associate to each generator $T_p = \{M_{AB}, Q_A\}$ a 1-form field $A^p = \{\omega^{AB}, \varphi^A\}$, and form a super Lie algebra valued connection 1-form,

$$A = A^p T_p = \omega^{AB} M_{AB} + \varphi^A Q_A, \tag{6}$$

where ω^{AB} is the SL(2,C) connection 1-form and φ^A is an anti-commuting spinor valued 1-form. (We shall use $\mathcal{D}$ for the Super SL(2,C) covariant derivative and D for the SL(2,C) covariant derivative.)

The curvature is given by $F = dA + \frac{1}{2}[A, A] = dA + \frac{1}{2} A^p \wedge A^q \otimes [T_p, T_q]$. Given the Super SL(2,C) connection A defined in equation (6), the curvature ($F = F(M)^{AB} M_{AB} + F(Q)^A Q_A$) contains a bosonic part associated with M_{AB},

$$F(M)^{AB} = d\omega^{AB} + \omega^{AC} \wedge \omega_C{}^B + \varphi^A \wedge \varphi^B; \tag{7}$$

and a fermionic part associated with Q_A ,

$$F(Q)^A = d\varphi^A + \omega^{AB} \wedge \varphi_B. \tag{8}$$

A simple *spinor* action, quadratic in the curvature, using this Super SL(2,C) connection A is

$$\begin{aligned}\mathcal{S}_{\mathrm{T}}[A^p] &= \int F^p \wedge F^q \, \eta_{pq} \\ &= \int F(M)^{AB} \wedge F(M)_{AB} \\ &\qquad +2F(Q)^A \wedge F(Q)_A,\end{aligned} \tag{9}$$

where η_{pq} is the Cartan-Killing metric of the Super SL(2,C) group, $\mathcal{D}\eta_{pq} = 0$. However, this action is a total differential. Hence, similar to the work of MacDowell and Mansouri [13], we need to choose another spinor action which is SL(2,C) invariant, thus breaking the topological field theory of the Super SL(2,C) symmetry into an SL(2,C) symmetry. Let us choose $i_{pq} = \mathrm{diag}(i_{(AB)(MN)}, i_{AB})$ such that

$$i_{(AB)(MN)} = \tfrac{1}{2}\left(\epsilon_{AM}\epsilon_{BN} + \epsilon_{AN}\epsilon_{BM}\right), \tag{10}$$

$$i_{AB} = 0. \tag{11}$$

The new spinor action is

$$\begin{aligned}\mathcal{S}[A^p] &= \int F^p \wedge F^q \, i_{pq} \\ &= \int F(M)^{AB} \wedge F(M)_{AB}.\end{aligned} \tag{12}$$

In terms of the SL(2,C) curvature $R^{AB} = d\omega^{AB} + \omega^{AC} \wedge \omega_C{}^B$, the spinor action $\mathcal{S}[A^p]$ is

$$\mathcal{S}[\omega^{AB}, \varphi^A] = \int R^{AB} \wedge R_{AB} + 2R_{AB} \wedge \varphi^A \wedge \varphi^B. \tag{13}$$

Note that, due to SL(2,C) Bianchi identity, the first term is purely topological, while the second term gives the dynamics. (The φ^4-term vanishes.) In the following discussion we consider only the dynamical term.

The field equations are obtained by varying the Lagrangian with respect to the gauge potentials—the Super SL(2,C) connection. With these gauge potentials fixed at the boundary, the field equations are

$$R^{AB} \wedge \varphi_B = 0 \qquad (DF(Q)^A = 0), \tag{14}$$

$$D(R^{AB} + \varphi^A \wedge \varphi^B) = 0 \qquad (DF(Q)^{AB} = 0), \tag{15}$$

where, becuase of the SL(2,C) Bianchi identidy ($DR^{AB}=0$), the second field equation (15) is reduced to $D(\varphi^A \wedge \varphi^B)=0$.

Another choice of the spinor action which is SL(2,C) invariant is that $i'_{pq} = \mathrm{diag}(i'_{(AB)(MN)}, i'_{AB})$ where

$$\begin{aligned} i'_{(AB)(MN)} &= 0, & (16)\\ i'_{AB} &= -2\epsilon_{AB}, & (17) \end{aligned}$$

then the new spinor action is

$$\begin{aligned} \mathcal{S}'[A^p] &= \int F^p \wedge F^q \, i'_{pq} \\ &= \int 2F(Q)^A \wedge F(Q)_A \\ &= \int 2D\varphi^A \wedge D\varphi_A. \end{aligned} \tag{18}$$

This action differs from the previous one by a total differential, a surface term:

$$\mathcal{S} = -\mathcal{S}' + \oint 2D\varphi^A \wedge \varphi_A. \tag{19}$$

Thus they give the same field equations.

In order to make a connection between the internal space of the Super SL(2,C) group with the structures on the four-manifold, we follow a construction in Kerrick [14]: we observe that although $\mathcal{D}\eta_{pq}=0$, $\mathcal{D}i_{pq}$ does not vanish, thereby providing a *soldering form* from the tangent space of the four-manifold to the internal space of Super SL(2,C).

Therefore, when we break the symmetry from a Super SL(2,C) topological field theory $\mathcal{S}_{\mathrm{T}}[A^p]$ into an SL(2,C) invariant $\mathcal{S}[A^p]$, a *spinor metric* is *naturally* defined. Using the fact that

$$\mathcal{D}i_{pq} = C^m{}_{pn} A^n i_{mq}, \tag{20}$$

and $\mathcal{D}\eta_{pq}=0$, the *spinor metric* $\mathcal{G}$ is defined by

$$\mathcal{G} = \eta^{pm}\eta^{qn}\mathcal{D}i_{pq} \otimes \mathcal{D}i_{mn} = \epsilon_{AB}\varphi^A \otimes \varphi^B. \tag{21}$$

Because $\mathcal{D}\eta_{pq} = \mathcal{D}i_{pq} + \mathcal{D}i'_{pq} = 0$, the *same* metric is defined when we choose the action $\mathcal{S}'$ with i_{pq} replaced by i'_{pq}.

For non-degenerate solutions ($\varphi^0 \wedge \varphi^1 \wedge \overline{\varphi}^{0'} \wedge \overline{\varphi}^{1'} \neq 0$) in which φ^0, φ^1 and their complex conjugates $\overline{\varphi}^{0'}, \overline{\varphi}^{1'}$ are linear independent, the 2-form $\varphi^A \wedge \varphi^B$ and their complex conjugates $\varphi^{A'} \wedge \varphi^{B'}$ form a basis for the six-dimensional space of 2-form. We can form a 1-1 map such that:

$$
\begin{aligned}
\varphi^0 \wedge \varphi^0 &\mapsto \theta^{00'} \wedge \theta^{01'}, && (22)\\
\varphi^0 \wedge \varphi^1 &\mapsto \tfrac{1}{2}(\theta^{00'} \wedge \theta^{11'} - \theta^{01'} \wedge \theta^{10'}), && (23)\\
\varphi^1 \wedge \varphi^1 &\mapsto \theta^{10'} \wedge \theta^{11'}, && (24)
\end{aligned}
$$

and their complex conjugates

$$
\begin{aligned}
\overline{\varphi}^{0'} \wedge \overline{\varphi}^{0'} &\mapsto \theta^{00'} \wedge \theta^{10'}, && (25)\\
\overline{\varphi}^{0'} \wedge \overline{\varphi}^{1'} &\mapsto \tfrac{1}{2}(\theta^{00'} \wedge \theta^{11'} + \theta^{01'} \wedge \theta^{10'}), && (26)\\
\overline{\varphi}^{1'} \wedge \overline{\varphi}^{1'} &\mapsto \theta^{01'} \wedge \theta^{11'}. && (27)
\end{aligned}
$$

This can be established by introducing an anti-commuting spinor field $\xi^{A'}$ such that $\bar{\kappa} = \epsilon_{A'B'}\xi^{A'}\xi^{B'}$ is a nilpotent constant. (This can be constructed in terms of an anti-commuting dyad where $\xi^{0'} = o^{0'}$, $\xi^{1'} = \iota^{1'}$ are odd constant spinors. They are spin basis with $o^{A'}o_{A'} = \iota^{A'}\iota_{A'} = 0$, $\iota^{0'} = o^{1'} = 0$, $o_{A'}\iota^{A'} = 2o^{0'}\iota^{1'} = \bar{\kappa}$, where $\bar{\kappa}$ is a nilpotent constant even element of the Grassmann algebra, see also Robinson[6].)

Now we relate the field φ^A to the tetrad field $\theta^{AA'}$ by $\theta^{AA'} = \varphi^A\xi^{A'}$, thus

$$\bar{\kappa}\,\varphi^A \wedge \varphi^B = \theta^{(AA'} \wedge \theta^{B)}{}_{A'}. \tag{28}$$

The *spacetime* metric is then defined by multiplying the *spinor* metric (21) by a nilpotent constant $\bar{\kappa}$,

$$g = \bar{\kappa} \otimes \mathcal{G} = \epsilon_{AB}\epsilon_{A'B'}\theta^{AA'} \otimes \theta^{BB'}, \tag{29}$$

which is the usual tetrad expression for spacetime metric.

Multiplying the spinor action (13) by $\bar{\kappa}$, we get

$$
\begin{aligned}
S[A^p, \bar{\kappa}] &= \bar{\kappa}\mathcal{S}[A^p]\\
&= \int 2R_{AB} \wedge \theta^{AA'} \wedge \theta^{B}{}_{A'}, && (30)
\end{aligned}
$$

the usual chiral action for general relativity .

By varying the action S with respect to $\theta^{AA'} = \varphi^A \xi^{A'}$, we obtained the chiral Einstein equation,

$$R^{AB} \wedge \varphi_B \xi_{B'} = R^{AB} \wedge \theta_{BB'} = 0. \tag{31}$$

Varying ω^{AB}, we obtained

$$\bar{\kappa} D(\varphi^A \wedge \varphi^B) = D(\theta^{(AC'} \wedge \theta^{B)}{}_{C'}) = 0, \tag{32}$$

which is the torsion free equation.

If we start with the action $\mathcal{S}'$, and multiply the spinor action by $\bar{\kappa}$, by using (19), we get

$$\begin{aligned} & S'[A^p, \bar{\kappa}] = \bar{\kappa}\mathcal{S}'[A^p] \\ = & \int \bar{\kappa}\left(-2R_{AB} \wedge \varphi^A \wedge \varphi^B + d(2D\varphi^A \wedge \varphi_A)\right) \\ = & \int -2R_{AB} \wedge \theta^{AA'} \wedge \theta^B{}_{A'} + d(2\bar{\kappa} D\theta^{AA'} \wedge \theta_{AA'}), \end{aligned} \tag{33}$$

where we used $d\bar{\kappa} = 0$. It reduces to the same action as S up to a sign and a total differential, and thus gives the same field equations.

The Hamiltonian can be constructed from either the action (13) or (18) by making a space-time decomposition

$$S = \bar{\kappa} \int dt \int d^3x \ \dot{A}_i^p \pi_p^i - \mathcal{H}. \tag{34}$$

For the action (13), the canonical momentum are:

$$\pi^{iA} = 0, \qquad \pi^{iAB} = \epsilon^{ijk} \varphi_j^A \varphi_k^B. \tag{35}$$

This yields the Hamiltonian density

$$\mathcal{H} = \omega_0^{AB} \mathcal{H}_{AB} + \varphi_0^A \mathcal{H}_A, \tag{36}$$

where

$$\mathcal{H}_{AB} = D_i \pi^i_{AB} = 0, \tag{37}$$

and

$$\mathcal{H}_A = \epsilon^{ijk} R_{ijAB} \varphi_k^B = 0, \tag{38}$$

are the constraints.

The canonical variables are related to the ones used in Ashtekar [2] by

$$\tilde{\sigma}^{iMN} = \bar{\kappa}\pi^{iMN} = \bar{\kappa}\epsilon^{ijk}\varphi_j^M\varphi_k^N, \tag{39}$$

$$q_{ij} = \bar{\kappa}\epsilon_{AB}\varphi_i^A\varphi_j^B, \tag{40}$$

where $\tilde{\sigma}^{iMN}$ is a densitized triad and q_{ij} is the 3 metric.

For the action (18), the canonical momentum are:

$$\pi^{iA} = \epsilon^{ijk}D_j\varphi_k^A, \qquad \pi^{iAB} = 0, \tag{41}$$

yielding a Hamiltonian density

$$\mathcal{H}' = \omega_0^{AB}\mathcal{H}'_{AB} + \varphi_0^A\mathcal{H}'_A + \partial_i(\varphi_0^A\pi_A^i), \tag{42}$$

where

$$\mathcal{H}'_{AB} = \varphi_{iB}\pi_A^i = 0, \tag{43}$$

and

$$\mathcal{H}'_A = D_i\pi_A^i = 0, \tag{44}$$

are the constraints.

The Hamiltonian (42) differs from (36) by a total differential. It has asymptotically flat fall off of $O(1/r^4)$, and the variation of the Hamiltonian will have an $O(1/r^3)$ boundary term which vanishes asymptotically. Therefore there is no need of adding a further boundary term for the Hamiltonian (42) [15]. With the field equations satisfied, the Hamiltonian (42) reduces to an exact differential. Integration yields an integral over a 2-surface

$$E = \bar{\kappa}\int \mathcal{H}' \mid_{\mathrm{S}} = \bar{\kappa}\oint \varphi_0^A D\varphi_A, \tag{45}$$

which determines the energy within a 2-surface.

This is comparable with the usual spinorial quasilocal energy constructions

$$E_w = \oint \xi_A D\xi_{A'} \wedge \theta^{AA'}, \tag{46}$$

with $\theta^{AA'} = \varphi^A\xi^{A'}$, it reduces to

$$E_w = \bar{\kappa}\oint \xi^A D\varphi_A. \tag{47}$$

By choosing $\varphi_0^A = \xi^A$, the energy expression (45) is related to the Witten expression E_w (47), while the spinors are now anti-commuting.

We can obtain a 4-component spinors formulation by introducing a Majorana spinor $\Psi = (\varphi_A, \overline{\varphi}^{A'})$ in the Weyl representation, the spinor action can be alternatively written as

$$\mathcal{S}_\Psi[\Psi^\alpha, \omega^{ab}] = \int \mathcal{L}_\Psi = 2\int \overline{D\Psi} \wedge \gamma_5 D\Psi. \tag{48}$$

The *real* SO(3,1) connection 1-form $\omega^{ab} = \omega^{AB}\epsilon^{A'B'} + \overline{\omega}^{A'B'}\epsilon^{AB}$ consists of the unprimed self-dual connection and its conjugate.

In the Weyl representation, the covariant derivative of the Dirac spinor 1-forms is given by

$$\overline{D\Psi} = (D\varphi^A \quad \overline{D}\overline{\varphi}_{A'}), \qquad D\Psi = \begin{pmatrix} D\varphi_A \\ \overline{D}\overline{\varphi}^{A'} \end{pmatrix}, \tag{49}$$

and $\gamma_5 := \gamma_0\gamma_1\gamma_2\gamma_3 = \mathrm{i}\ \mathrm{diag}(-1,-1,1,1)$. The Lagrangian in (48) can thus be split into unprimed and primed parts:

$$\mathcal{L}_\Psi = 2\mathrm{i}\ D\varphi^A \wedge D\varphi_A - 2\mathrm{i}\ \overline{D}\overline{\varphi}^{A'} \wedge \overline{D}\overline{\varphi}_{A'}, \tag{50}$$

where the second term is just the conjugate of the first term. The action $\mathcal{S}_\Psi$ in (48) is therefore just the real part of the action $\mathcal{S}'$ in (18).

Now a key observation is that $\mathcal{S}[z]$ is a holomorphic function of z, where $z := (\varphi^A, \omega^{AB})$. Therefore, just as for analytic functions of a finite number of variables, the derivative $\delta\mathcal{S}[z]/\delta z$ vanishes if and only if the derivative of the real (or imaginary) part of $\mathcal{S}[z]$ vanishes. Thus, as far as the equations of motion are concerned, the action $\mathcal{S}_\Psi$ (48) is *equivalent* to the original chiral action $\mathcal{S}$.

In conclsion, we proposed a gauge theory for the anticommuting version of the quadratic spinor Lagrangian for general relativity. The gauge potential is a connection of the Super SL(2,C) group. The action is quadratic in the Super SL(2,C) curvature and depends purely on gauge connection. By breaking the symmetry of the Super SL(2,C) topological gauge theory to SL(2,C), a spinor metric is given and is related to the spacetime metric. In the Hamiltonian formulation, the canonical momentum π^i_{AB} conjugates to the SL(2,C) connection ω_i^{AB} is related to the triad in Ashtekar's New Variables. Because of the nature of using a Majorana spinor 1-form field,

π^i_{AB} is allowed to be complex while keeping the right degrees of freedom. The reality condition is replaced by a Majorana condition. This suggests a question: "What are the local properties of gravitational field?" Among other properties, this letter suggests that the Majorana property might be one of the local properties for gravitation, which means that the resulting particle is neutral so that it contains no charge.

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References

[1] A. Trautman, in *General Relativity and Gravitation*, vol. I, ed. A. Held (Plenum, New York, 1980); J. M. Nester, in *An Introduction to Kaluza-Klein Theories*, ed. H. C. Lee (World Scientific, Singapore, 1984).

[2] A. Ashtekar, *Phys. Rev. Lett.* **57** 2244 (1986); *Phys. Rev. D* **36** 1587 (1987).

[3] R. Capovilla, J. Dell and T. Jacobson, *Phys. Rev. Lett.* **63**, 2325 (1989).

[4] J. M. Nester and R. S. Tung, *Gen. Rel. Grav.* **27** , 115 (1995).

[5] R. S. Tung and T. Jacobson, *Class. Quantum Grav.* **12**, L51 (1995).

[6] D. C. Robinson, *Class. Quantum Grav.* **12**, 307 (1996).

[7] R. S. Tung and J. M. Nester, *Phys. Rev. D* **60**, 021501 (1999).

[8] D. C. Robinson, *Int. J. Theor. Phys.* **37**, 2067 (1998).

[9] T. Jacobson, *Private Communication.*

[10] I. Bars and S. W. MacDowell, *Phys. Lett. B* **71**, 111 (1977); *Gen. Rel. Grav.* **10**, 205 (1979).

[11] R. Penrose and W. Rindler, *Spinors and Space-Time* Vol.1 (Cambridge: Cambridge University Press, 1984).

[12] B. DeWitt, *Supermanifolds.* (Cambridge: Cambridge University Press, 1984).

[13] S. W. MacDowell and F. Mansouri, *Phys. Rev. Lett.* **38**, 739 (1977); **38**, 1376 (1977).

[14] D. M. Kerrick, *Phys. Rev. Lett.* **75**, 2074 (1995).

[15] T. Regge and C. Teitelboim *Ann. Phys.* (NY) **88**, 286 (1974).

[16] J. A. Nieto, J. Socorro and O. Obregón, *Phys. Rev. Lett.* **76**, 3482 (1996).

[17] L. Smolin, *Nucl. Phys. B* **148**, 333 (1979).